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Time-like T-duality algebra

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Abstract: When compactifying M- or type II string-theories on tori of indefinite space-time signature, their low energy theories involve sigma models on $E_{n(n)}/H_n$, where H_n is a not necessarily compact subgroup of $E_{n(n)}$ whose complexification is identical to the complexification of the maximal compact subgroup of $E_{n(n)}$. We discuss how to compute the group H_n. For finite dimensional $E_{n(n)}$, a formula derived from the theory of real forms of E_n algebra's gives the possible groups immediately. A few groups that have not appeared in the literature are found. For $n = 9, 10, 11$ we compute and describe the relevant real forms of E_n and H_n. A given H_n can correspond to multiple signatures for the compact torus. We compute the groups H_n for all compactifications of M-, M^*-, and M'-theories, and type II- II^*- and II'- theories on tori of arbitrary signature, and collect them in tables that outline the dualities between them. In an appendix we list cosets G/H, with G split and H a subgroup of G, that are relevant to timelike toroidal compactifications and oxidation of theories with enhanced symmetries.

Keywords: M-Theory, String Duality, Space-Time Symmetries.

Contents

1. Introduction

The advent and understanding of duality symmetries has radically changed our view of high-energy physics. Such symmetries transform excitations of elementary fields into solitons, can change the topology of space-time, and allows to interpret seemingly very different theories, as different expansions from a single underlying theory [1, 2].

Under duality symmetries not much is sacred, and it was found that there are even duality symmetries that can change the signature of space-time [3, 4, 5]. To establish such a duality, one compactifies the theory on a time-like circle. There are however a number of subtleties with time-like compactifications.

It has been known for a long time that toroidal compactification of 11 dimensional supergravity [6] results in effective theories with the scalars organized in sigma models on symmetric spaces of the form $E_{n(n)}/H_n$ [7, 8, 9] where $E_{n(n)}$ is the split (maximal non-compact) real form of an exceptional Lie-algebra, and H_n is its maximal compact subgroup. The number n equals $11-d$ for a compactification to d dimensions. For $d \geq 3$ the global symmetry $E_{n(n)}$ has been checked in detail in [10], which for $d=4$ was already established in [7] (see [11] for a review and more references). For $0 < d < 3$ the symmetries are supported by considerable evidence [9, 12, 13, 14, 15].

Applying the reduction program to include time-like directions, entails a number of modifications. The coset symmetries found still have scalars on cosets $E_{n(n)}/H_n$, but although the denominator group $E_{n(n)}$ is still the split real form of the exceptional E_n group, the groups H_n are no longer compact [16, 17]. Instead the groups H_n have the same complexification as the maximal compact subgroup, but typically are non-compact real forms. The papers [16, 17] considered dimensional reduction of conventional supergravities (in a space-time with 1 time-direction) with *one* time-direction included, yet the arguments of [3, 4, 5] imply the existence of less conventional supergravities formulated on space-times with more than one time-like direction.

In this paper, we study all these variant supergravities, reduced over tori of arbitrary space-time signature. There are many possibilities (many supergravity variants, many possibilities for the space-time signature), and it is actually more convenient to study the problem from the other end. The global symmetry groups $E_{n(n)}$ are constant elements in the discussion, so we need to determine which groups H_n can appear as denominator subgroups, and, how these groups relate to the space-time signature of the compact torus, as well as the various signs for the form-field terms in the lagrangian that distinguish various supergravities from one another.

This is possible using extensions of the techniques from [18]. In this paper the implications of space-time signature in the context of the E_{11}-conjecture [19] were studied. The E_{11}-conjecture states that there exists a hypothetical formulation of (the bosonic sector of) *11-dimensional* supergravity/M-theory, with a non-linearly realized E_{11} symmetry. The variables describing the theory are specified by the coset $E_{11(11)}/H_{11}$, where H_{11} is a subgroup of E_{11}. This subgroup is necessarily non-compact, as it is proposed that H_{11} contains the Lorentz-group $\mathrm{SO}(1,10)$ for 11 dimensions. The paper [18] demonstrates that there are also other real forms of $\mathrm{SO}(11,\mathbb{C})$ contained in H_{11}. The possibility to select these as Lorentz group implies that a formalism based on a non-linearly realized E_{11} symmetry (if true) describes not only conventional 11 dimensional supergravity, but also theories in other space-time signatures. In this way it makes contact with the results of [3, 4], and indeed one can show that the E_{11}-conjecture has the potential to describe all the theories in these papers.

The techniques that were applied in [18] to $E_{11(11)}$ and H_{11} apply equally well to $E_{n(n)}$ and H_n for $n \neq 11$ (and in fact to any Lie group G). Two crucial elements in the discussion of [18] are $\mathbb{Z}_2$-valued functions on the root lattice and Weyl-reflections. As we will explain in this paper, the classification of $\mathbb{Z}_2$-valued functions amounts to nothing but the classification of inner involutions on the E_n algebra. For the finite dimensional E_n this has been established long ago, in the context of the classification of real forms of the algebra. Working out this connection, we obtain a simple formula which immediately gives all the possible denominator subgroups. Among these there are a few that have not appeared in [16, 17, 3, 4]. For infinite-dimensional E_n and H_n our techniques fix the real forms that are possible in principle, and those that actually do appear in compactifications of maximal supergravities (just as in the E_{11} case, a number of real forms is ruled out by the requirement that they have to connect to conventional 11 dimensional supergravity [18]).

Weyl reflections are instrumental. Establishing the global and local symmetries in supergravity, it is conventional to choose a particular realization of the symmetry, with a one-to-one correspondence between the basis of the Cartan subalgebra of $E_{n(n)}$ and the dilatonic scalars in the theory (see e.g. [10, 20, 21, 22]). The roots of the algebra, and their associated operators then correspond to axions and their duals. In this form the algebra is essentially fixed, and there are no continuous E_n transformations possible anymore. There is however a discrete set of E_n transformations that rotate the root lattice into itself; these are the Weyl-reflections. As we will recall in subsection 3.1 these Weyl-reflections correspond to nothing but a particular subset of the duality transformations in the theory; this implies that the action of the Weyl-group is closely linked to the group of T-dualities, including the time-like ones.

The set-up of this paper is as follows. In section 2 we set the stage for the mathematical formalism and introduce our conventions. We then explain how our problem is linked to the theory of real forms of algebra's, culminating in equation (2.28) which is one of our core results. Section 3 explains how to perform explicit computations, identify dualities, extract space-time signatures and other signs. Section 4 applies the formalism to compactifications of M-theory to $d \geq 3$ dimensions, where we can use the link to real forms of E_n algebra's to shortcut a number of computations. Section 5 deals with the same problem, but now for $d < 3$. As the mathematical groups appearing here are less familiar, we devote some more space to discussion of their properties. Section 6 deals with the (relatively trivial) link to IIA-string and supergravity theories. Section 7 studies an alternative embedding of the Lorentz-algebra of the torus, resulting in IIB-theories and their variants. The sections 4, 5, 7 contain many tables that should be helpful to the reader who is interested in the dualities and the cosets, but not necessarily in the full machinery behind them. Finally, in section 8, we summarize our results. We have added an appendix with a table containing

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the groups related by our equation (2.28) for cosets with a split Lie-group in the numerator; these are relevant for time-like reduction and oxidation of theories described in [20, 21].

2. Group theory

2.1 Definition and properties of the E-algebra's

In this section we recall some facts about the general theory of Kac-Moody algebra's [23], and the E_n algebra's in particular. Our conventions are chosen such that for $n = 11$ we recover the conventions of the paper [18], apart from the fact that here we order the nodes along the horizontal line in the Dynkin diagram in the opposite direction.

We start by drawing the Dynkin diagrams for E_n.

For $n < 6$ there exist alternative names for these algebra's, as $E_3 \cong A_1 \oplus A_2$, $E_4 \cong A_4$, $E_5 \cong D_5$.

From the Dynkin diagram the Cartan matrix $A_n = (a_{ij})$, with i, j in the index set $I \equiv \{0, 1, \ldots, n-1\}$, may be reconstructed by setting

$$a_{ij} \equiv \begin{cases} \;\;2 \text{ if } i = j\,; \\ -1 \text{ if } i, j \text{ connected by a line}\,; \\ \;\;0 \text{ otherwise}\,. \end{cases} \tag{2.1}$$

The Cartan matrix is symmetric, and $\det(A_n) = 9 - n$. For $n \neq 9$, we choose a real vector space $\mathcal{H}$ of dimension n and linearly independent sets $\Pi = \{\alpha_0, \ldots, \alpha_n\} \subset \mathcal{H}^*$ (with $\mathcal{H}^*$ the space dual to $\mathcal{H}$) and $\Pi^\vee = \{\alpha_0^\vee, \ldots, \alpha_n^\vee\} \subset \mathcal{H}$, obeying $a_{ij} = \alpha_j(\alpha_i^\vee) \equiv \langle \alpha_j, \alpha_i^\vee \rangle$. The elements of the set Π are called the simple roots. Linear combinations of the elements of Π with integer coefficients span a lattice, called the root lattice of E_n, which we will denote by P_n. Similarly, the $\alpha_i^\vee$ also span a lattice $P_n^\vee$ known as the coroot lattice.

For the affine Lie-algebra E_9 one has $\det(A_9) = 0$, and the construction of the full algebra requires one to introduce a vector space $\mathcal{H}$ of dimension 10 (instead of 9). The Cartan sub-algebra has one extra generator, the central charge K. This special feature is not playing any role in our discussion, as eventually we will be interested in the subalgebra H_9 of E_9, of which K is not a generator. We therefore proceed, and refer to [23] for details on affine algebra's.

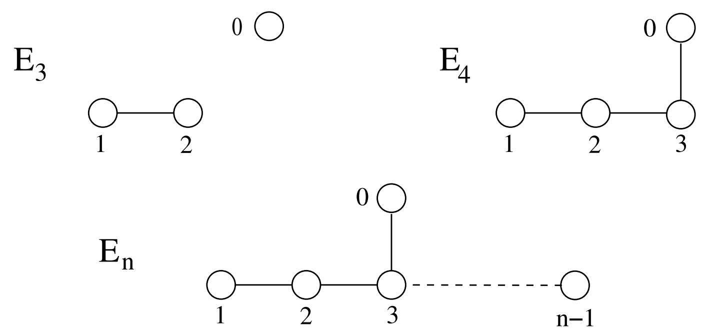

Figure 1: The Dynkin diagrams of E_n algebra's.

From the Cartan matrix the algebra E_n can be constructed. The generators of the algebra consist of 11 basis elements h_i for the Cartan sub algebra $\mathcal{H}$ together with 22 generators e_{α_i} and $e_{-\alpha_i}$ ($i \in I$), and of algebra elements obtained by taking multiple commutators of these. These commutators are restricted by the algebraic relations (with $h, h' \in \mathcal{H}$):

$$[h, h'] = 0 \qquad [h, e_{\alpha_j}] = \langle\alpha_j, h\rangle e_{\alpha_j} \qquad [h, e_{-\alpha_j}] = -\langle\alpha_j, h\rangle e_{-\alpha_j} \qquad [e_{\alpha_i}, e_{-\alpha_j}] = \delta_{ij}\alpha_i^\vee\,; \tag{2.2}$$

and the Serre relations

$$\mathrm{ad}(e_{\alpha_i})^{1-a_{ij}} e_{\alpha_j} = 0\,, \qquad \mathrm{ad}(e_{-\alpha_i})^{1-a_{ij}} e_{-\alpha_j} = 0\,. \tag{2.3}$$

Although in the theories we will be considering the relevant real form of E_n, is the so-called split real form, other real forms play an auxiliary role in our computations. We will discuss real forms in subsection 2.3.

There is a natural bijection $\alpha_i \to \alpha_i^\vee$, in which the components of each $\alpha_i^\vee$ turn out to be identical to those of α_i (This is because the E_n are simply-laced algebra's). To some extent, one may regard the $\alpha_i^\vee$ as "column vectors", and the α_i as "row vectors". Under the bijection, the coroot lattice, consisting of linear combinations with integer coefficients of $\alpha_i^\vee$ may therefore be identified with the root lattice.

Under the root space decomposition with respect to the Cartan subalgebra, E_n is decomposed into subspaces of eigenvectors with respect to the adjoint action of the Cartan sub-algebra. Schematically, $E_n = \oplus_{\alpha\in\mathcal{H}^*} g_\alpha$, with

$$g_\alpha = \{x \in E_n : [h, x] = \langle\alpha, h\rangle x \;\forall h \in \mathcal{H}\} \tag{2.4}$$

The set of roots of the algebra, Δ, are defined by

$$\Delta = \{\alpha \in \mathcal{H} : g_\alpha \neq 0, \alpha \neq 0\} \tag{2.5}$$

The roots of the algebra form a subset of the root lattice $\Delta \subset P_n$. The set of positive roots $\Delta^+ \subset \Delta$ is the subset of roots whose expansion in the simple roots involves non-negative integer coefficients only. We denote the basis elements of g_α by e_α^k, where k is a degeneracy index, taking values in $\{1, \ldots, \dim(g_\alpha)\}$. If $\dim(g_\alpha) = 1$, we will drop the degeneracy index, and write e_α for the generator. Note that previously we defined the generators $e_{\pm\alpha_i}$ for α_i a simple root, and as $\dim(g_{\alpha_i}) = 1$ when α_i is a simple root, this is consistent with the conventions we have laid out here. With the aid of the Jacobi identity, it is easily proven that $[e_\alpha^i, e_\beta^j] \in g_{\alpha+\beta}$, if this commutator is different from zero.

Another aspect that will enter prominently into the discussion are the hermiticity properties of the operators e_α^k and h. In our conventions these will be chosen as

$$h^\dagger = h \;\forall\; h \in \mathcal{H} \qquad \left(e_\alpha^k\right)^\dagger = e_{-\alpha}^k\,. \tag{2.6}$$

For the specific E_n root systems we recall that the set Δ consists of finitely many elements for E_n with $n < 9$, and of infinitely many elements if $n \geq 9$. The Cartan matrix

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of E_n implies that the inner product on the root space as we have defined it is of positive definite signature for $n < 9$, contains one null-direction if $n = 9$, and is of lorentzian signature (i.e. $(1, n-1)$) for $n > 9$.

The lattice dual to the root lattice is called the coweight lattice, which we will denote as Q_n. If we define the fundamental coweights by

$$\langle \alpha_i, \omega_j \rangle = \delta_{ij} , \qquad \alpha_i \in \Pi \tag{2.7}$$

then we can describe Q_n as generated by linear combinations of the fundamental coweights with coefficients in $\mathbb{Z}$. It is clear that the coweight lattice contains the coroot lattice $P_n^\vee$, as a sublattice.

We will make much use of the Weyl group W_n of E_n. This is the group generated by the Weyl reflections w_i in the simple roots,

$$w_i(\beta) = \beta - \langle \beta, \alpha_i^\vee \rangle \alpha_i . \tag{2.8}$$

The Weyl group leaves the inner product invariant

$$\langle w(\alpha), w(\beta) \rangle = \langle \alpha, \beta \rangle \qquad w \in W_n . \tag{2.9}$$

The Weyl group includes reflections in the non-simple roots.

2.2 The subalgebra H_n

In the relevant, dimensionally reduced or unreduced theories, the E_n algebra's are not manifest, but non-linearly realized. The relevant variables to describe the theory are captured by the coset E_n/H_n, where H_n is a subgroup of E_n. Now in the original formulation of the problem, these cosets appeared in the dimensional reduction over space-like directions only, of the 11 dimensional supergravity in space-time signature (1,10). In that case, the groups H_n are compact.

Recalling that the compact subgroup of H_n is generated by the generators $e^k_\alpha - e^k_{-\alpha}$, it was proposed in [24] to modify these to take into account non-compact generators (we will present a more precise definition in the next subsection). The generators of H_n are then taken to be

$$T^k_\alpha = e^k_\alpha - \epsilon_\alpha e^k_{-\alpha} . \tag{2.10}$$

As in [18], the ϵ_α cannot depend on k, as we will review in a moment.

Also in [18], it was argued that in spite of the infinitely big root system of E_{11}, it is actually sufficient to specify the ϵ_α for a basis of simple roots. The argument does not depend on the fact that the underlying algebra is E_{11}, and we will briefly repeat the relevant steps here.

The first step in the argument consists of noting that, due to equation (2.6),

$$(T^k_\alpha)^\dagger = -\epsilon_\alpha T^k_\alpha , \tag{2.11}$$

such that the sign ϵ_α actually encodes the hermiticity properties of the generator T^k_α.

Then we note that the generators $T^k_{\alpha\pm\beta}$ actually appear in the commutator of T^k_α and T^k_β. This, together with the reality of the structure constants of E_n, immediately implies that the hermiticity properties of $T^k_{\alpha\pm\beta}$ follow from those of T^k_α and T^k_β.

Then we realize that any generator T^k_α can be formed by taking multiple commutators of the T_{α_i}, where the α_i are the simple roots. From this it follows immediately that we were correct in asserting that specifying ϵ_{α_i} for the simple roots α_i, we have fixed and described the non-compact form of the algebra completely. Moreover, it is easy to see that the coefficients ϵ_α depend on α, but not on the degeneracy index k.

This information is now transferred to a function $f(\alpha)$, that is related to ϵ_α by

$$\epsilon_\alpha = \exp(i\pi f(\alpha))\,. \tag{2.12}$$

Because of the properties of the commutator, it follows immediately that

$$f(\alpha+\beta) = f(\alpha) + f(\beta) \tag{2.13}$$

and hence f is a linear function, taking values in $\mathbb{Z}_2$. Note that the minus sign in (2.11) is crucial in establishing linearity.

We furthermore note that all such functions are described by

$$f(\alpha) = \sum_i p_i \langle \alpha, \omega_i \rangle \tag{2.14}$$

where ω_i are the fundamental coweights, and the p_i are coefficients in $\mathbb{Z}_2$. Different choices for the Dynkin basis of the algebra are related by the elements of the Weyl group, and we should therefore not distinguish between f and its images generated by Weyl reflections. The physical interpretation relies on the intersection of an A_n-algebra with the full algebra and this may change under Weyl reflections. This is because the non-trivial Weyl reflection (the ones that cannot be interpreted as permutations of coordinates) correspond to sequences of T-dualities, as we will review in section 3.1. We can now proceed as in [18], but there is actually one more notable point.

In terms of the complexification $(E_n)^{\mathbb{C}}$ of the algebra, we have

$$\exp(i\pi f) h \exp(-i\pi f) = h \tag{2.15}$$

$$\exp(i\pi f) e_\beta \exp(-i\pi f) = \exp(i\pi f(\beta)) e_\beta\,. \tag{2.16}$$

Note that we have to turn to $(E_n)^{\mathbb{C}}$, because $\exp(i\pi f)$ is not an element of the the split real form as we have defined it in the above. These relations mean that $\exp i\pi f$, via the adjoint action, defines an involution of the algebra $(E_n)^{\mathbb{C}}$. Because $\exp i\pi f$ represents a group element (of the complex group), this involution is *inner*. Conversely, if an involution is inner, it is conjugate to an element of the form $\exp(i\pi f)$, which is on the maximal torus (the exponentiation of the Cartan sub-algebra). The algebraic characterization them implies that $f \in Q_n$.

2.3 Involutions and real forms

The result of the previous section essentially implies that we are looking for those involutions on the algebra E_n that are inner. In Lie algebra theory, the study of involutions is tightly connected to the study of real forms of Lie algebra's, and we will actually borrow some results from there.

The central object in the study of real forms of semi-simple Lie-groups is that of the Cartan involution [25]. From the Cartan involution the non-compact real form of the group can be easily reconstructed.

An involutive automorphism θ is called a Cartan involution if $-\langle X, \theta Y\rangle$ is strictly positive definite for all algebra generators X, Y. An involution has by definition eigenvalues ± 1 and the realization of the involution can be chosen such that the Cartan subalgebra is closed under the involution.

In the supergravity literature the Cartan involution is usually chosen to be realized in the way that is encoded in a Satake diagram (see [25] for mathematical background, or [22] for a physicists account of the theory with applications to gravity and supergravity). This way of realizing the involution can be adapted to all possible real forms, but we will need it here only to realize the so-called split real form of the algebra. We call the split real form $E_{n(n)}$, as usual, and realize it by using a Cartan involution acting on the root space as

$$\theta(\alpha) = -\alpha\,. \tag{2.17}$$

The corresponding generators of the real form of the algebra are $h \in \mathcal{H}$, $e^k_\alpha + e^k_{-\alpha}$ and $e^k_\alpha - e^k_{-\alpha}$ with $\alpha \in \Delta$ (we have anticipated our application to infinite dimensional algebra's, where $\dim(g_\alpha)$ can be bigger than one and we will need the degeneracy index k). This implies that all generators of E_n can be formed by linear combinations of elements h, $e_{\pm\alpha}$ with *real* coefficients.

For the definition of the denominator sub-algebra H_n we require only those involutions that are inner. These inner involutions are specified by a function f, and we will now work out the realization of the real form defined by f.

This real form, that we will call $\mathcal{E}_n$, defined by the involution exhibited in (2.15) and (2.16) has as its generators

$$ih \qquad\qquad \forall\, h \in \mathcal{H} \tag{2.18}$$

$$e^k_\alpha - \exp(i\pi f(\alpha))e^k_{-\alpha} \tag{2.19}$$

$$i\left(e^k_\alpha + \exp(i\pi f(\alpha))e^k_{-\alpha}\right). \tag{2.20}$$

Obviously, in our conventions the generator (2.18) is anti-hermitian. The hermiticity properties of the generators (2.19) and (2.20) however depend on the value of the function $\exp i\pi f(\alpha) = \epsilon_\alpha$.

We can now define the group H_n in the following way. We have the complexified algebra $(E_n)^{\mathbb{C}}$ and have defined two real forms , both embedded in $(E_n)^{\mathbb{C}}$. One is the split real form, denoted as $E_{n(n)}$, and realized as described below (2.17). There is a second real form, not necessarily equivalent to the split real form, that we realize as in the equations (2.18), (2.19) and (2.20). This second real form we denote by $\mathcal{E}_n$, and it will play only an auxiliary (but crucial) role. We can now define the denominator subalgebra H_n as the intersection $\mathcal{E}_n \cap E_{n(n)}$. In a diagram:

$$\begin{array}{ccl} (E_n)^{\mathbb{C}} & \supset & \mathcal{E}_n \\ \cup & & \cup \\ E_{n(n)} & \supset & H_n = \mathcal{E}_n \cap E_{n(n)} \end{array} \tag{2.21}$$

With this explicit realization H_n is generated by generators of the form (2.19), which again are precisely generators of the form of equation (2.10) that were previously used to define the denominator subgroup H_n ad hoc. The relevant groups for supergravity theory are found on the lower line of (2.21) as the coset appearing will be $E_{n(n)}/H_n$. The groups at the upper line are useful for mathematical exploration, and in particular we will derive a nice result from the relation to the group $\mathcal{E}_n$.

Some remarks are in order. First, it is obvious that the diagram (2.21) can be generalized to other groups than the E_n series, and these should play a role in time-like compactification of the theories in [26, 20, 21, 22].

Second, the discussion here makes precise the meaning of the phrase "temporal involution", introduced in [24]. This involution is just an involution on the complex algebra $(E_n)^{\mathbb{C}}$, defining the real form $\mathcal{E}_n$, although one uses in effect only its restriction to H_n. An important thing to notice is that all (inner) involutions on $(E_n)^{\mathbb{C}}$ descend to involutions on H_n, but the reverse is not true. Hence, there may exist real forms of H_n that can not appear in the denominator of $E_{n(n)}/H_n$ in the context of (super-)gravity theories. This is related to the fact that in all studies, embeddings of subalgebra's are (sometimes implicitly) assumed to have certain regularity properties [8, 19]. An attempt at a motivation of the necessity of regularity of sub-algebra's in the context of (super-) gravity, for finite dimensional algebra's can be found in [21].

A third remark concerns the fact that in the supergravity literature one occasionally encounters the phrase "real forms of supergravities", referring to the various variant supergravities. Although there are many links with the theories of real forms of the underlying algebra's, as the present discussion and [18] demonstrate, it is not quite appropriate to refer to "real forms of supergravities". One obvious reason is that *different* variant supergravities are described by the *same* real form of the algebra, see [18] and the rest of this paper. We prefer therefore the more neutral "variant" above "real form", when referring to supergravity.

2.4 The signature of finite dimensional H_n

We can divide the algebra into hermitian generators, generating non-compact symmetries, and anti-hermitian generators, generating compact symmetries. If $\mathcal{G}$ is a finite dimensional real form, we can define $n(\mathcal{G})$ to be the number of non-compact generators of $\mathcal{G}$, and $c(\mathcal{G})$ the number of compact generators. For these finite dimensional algebra's one obviously has

$$\dim(\mathcal{G}) = n(\mathcal{G}) + c(\mathcal{G})\,. \tag{2.22}$$

The signature or character $\sigma(\mathcal{G})$ of $\mathcal{G}$ is defined to be the difference of these two quantities:

$$\sigma(\mathcal{G}) = n(\mathcal{G}) - c(\mathcal{G})\,. \tag{2.23}$$

We furthermore will need the rank $r(\mathcal{G})$ which is the dimension of the largest completely reducible abelian sub-algebra one can find.

Now we look at the inner involution that defines the real form for $\mathcal{E}_n$, as well as the algebra H_n. Some properties of $\mathcal{E}_n$ and H_n are easily related.

First, the non-compact elements of $\mathcal{E}_n$ are in one to one correspondence with those roots of E_n on which the involution has value -1. Due to linearity and the mod 2 property,

$$\exp(i\pi f(\alpha)) = \exp(i\pi f(-\alpha))\,. \tag{2.24}$$

Therefore positive and negative roots are paired by the involution, but actually the definition of the generators (2.19) and (2.20) imply that non-compact generators also always come in pairs. The Cartan generators commute with the maximal torus, and hence all correspond to compact generators, as is explicit in (2.18). The non compact elements of $H_n = \mathcal{E}_n \cap E_{n(n)}$ are in one-to-one correspondence with the *positive* roots on which the involution has value -1; only the generators (2.19) are contained in the intersection. Hence

$$n(H_n) = \frac{n(\mathcal{E}_n)}{2}\,. \tag{2.25}$$

The number of compact elements of H_n are given by

$$c(H_n) = \frac{c(\mathcal{E}_n) - r(\mathcal{E}_n)}{2} \tag{2.26}$$

where we have used that for the split real form, the dimension of H_n is equal to the number of positive roots of E_n, that can be computed from

$$\dim(H_n) = \frac{\dim(\mathcal{E}_n) - r(\mathcal{E}_n)}{2}\,. \tag{2.27}$$

Subtracting (2.26) from (2.25) and using the definitions, this little algebra reveals

$$\sigma(H_n) = \frac{\sigma(\mathcal{E}_n) + r(\mathcal{E}_n)}{2}\,. \tag{2.28}$$

Given a real form of $\mathcal{E}_n$, defined by an inner involution, with given signature $\sigma(\mathcal{E}_n)$ (these can be looked up in tables e.g. in [25], or for example in [21]), we can immediately, and trivially compute the signature for a particular possibility for H_n. In many instances, the fact that we know the compact form of the algebra H_n, together with the signature $\sigma(H_n)$ determines the real form of H_n completely. In the computations in the rest of this article there is only a single exception to this rule, to be discussed in subsection 4.4.

Equation (2.28) is one of the core results of our paper. It is easily seen that it generalizes to all instances of oxidation and dimensional reduction, that yield a coset sigma model on G/H with G a split real form. We have computed the list of all possible denominator subgroups H for a given split Lie-group G, and added it as appendix A to this paper. It should also be obvious how it generalizes to non-split G, although we have not found a formula generalizing (2.28) that is as compact and elegant.

3. Methods of computation

In this section we discuss a methods of computation. First however we recall an argument on why the methods based on Weyl-reflections should actually give us the correct signatures for time-like T-dualities.

3.1 The E_n Weyl group as the "self-duality group" for type-II strings

This section presents a variant on an argument that can be found in [11]. We believe it demonstrates the applicability of our results, even for those cases where we have to rely on conjectural symmetries.

In [11] it is shown that the E_n Weyl-group arises as a combination of geometrical symmetries with T-duality. It is instructive to reformulate the geometrical symmetries in terms of the S-duality of the type IIB superstring, as this will clarify the precise map between those symmetries that map a type-II-theory to itself, to the Weyl group of E_n.

Consider a type II theory compactified on an n-torus. The space-time signature of this torus is not relevant in this section.

The IIB-theory has an S-duality symmetry acting on the string coupling g and the radii of the compact directions R_i as:

$$\begin{aligned} \ln g &\to -\ln g \\ \ln R_i &\to \ln R_i - \frac{1}{2}\ln g \,. \end{aligned} \tag{3.1}$$

There are also T-duality symmetries mapping the IIA-theory to the IIB-theory and vice versa. The T-duality in the j-direction acts as

$$\begin{aligned} \ln g &\to \ln g - \ln R_i \\ \ln R_j &\to \ln R_j - 2\delta_{ij}(\ln R_i) \,. \end{aligned} \tag{3.2}$$

Given these transformations, we define for IIA-theory compactified on an n-torus the transformations W_j with $j \leq n$, and W_0 for $n \geq 2$:

$$W_1 = T_1 S T_1 \tag{3.3}$$

$$W_{i+1} = T_i S T_i T_{i+1} S T_{i+1} T_i S T_i = T_{i+1} S T_{i+1} T_i S T_i T_{i+1} S T_{i+1} \qquad i > 1 \tag{3.4}$$

$$W_0 = T_1 S T_1 T_2 S T_2 T_1 S T_2 \,. \tag{3.5}$$

The transformation W_1 is well-known as the transformation that exchanges the 11^{th} dimension in M-theory with the 1-direction of the IIA-theory. With this in mind it is also easily shown that the sequence of dualities described by W_{i+1} lead to nothing but a permutation of the i^{th} and $(i+1)^{th}$ direction. Note also that the elements of the Weyl group that change the orientation on the root lattice (and in particular, the reflections) have an odd number of S-entries.

These transformations map the IIA-string to itself, or to a variant of itself. It is easily verified that

$$(W_i)^2 = 1 \tag{3.6}$$

and if $i \neq j$

$$(W_i W_j)^{n_{ij}} = 1 \qquad n_{ij} = \begin{cases} 2\,, & \text{if } a_{ij}a_{ji} = 0 \\ 3\,, & \text{if } a_{ij}a_{ji} = 1 \end{cases} \tag{3.7}$$

where a_{ij} is an entry in the Cartan matrix A_n of E_n. This means that the group generated by the above transformations is a *Coxeter group*, and that moreover this Coxeter group is isomorphic to the Weyl group of E_n. This is actually well-known, but there are a few advantages to formulating the group this way.

First of all, we observe that we can form all the elements

$$T_1 T_i = \begin{cases} W_0 W_2\,, & i = 2\,; \\ W_i T_1 T_{i-1} W_i\,, & i > 2\,. \end{cases} \tag{3.8}$$

Together with $W_1 = T_1 S T_1$, we can form any element $T_i T_j$, and $T_i S T_j$. These in turn can be used as building blocks to build any duality transformation that maps the IIA-theory to itself (or a variant in a different space-time signature, and/or other signs in front of the form-field terms). This proves that the Weyl group of E_n is isomorphic to the group of "self-duality" transformations, and provides a one-to-one map between Weyl-group elements and duality transformations, by the identification of generators

$$W_i \leftrightarrow w_i\,. \tag{3.9}$$

The argument is translated to IIB theory by performing a T-duality transformation. It is most convenient to use a T-duality in the 1 direction for this, to obtain as generators

$$W_1 = S \tag{3.10}$$
$$W_2 = S T_1 T_2 S T_1 T_2 S \tag{3.11}$$
$$W_{i+1} = (S T_i T_{i+1})^3 \qquad i > 2 \tag{3.12}$$
$$W_0 = (S T_1 T_2)^3\,. \tag{3.13}$$

Again the combinations of two T-dualities can be built as in equation (3.8); together with the transformation $W_1 = S$ any duality transformation mapping type IIB theory to itself can be built from these.

This alternative view on Weyl group symmetries, which are an obvious consequence of the dimensionally reduced low energy theory, implies that these symmetries should lift to symmetries of the full (not-truncated) string- and M-theory (if T- and S-dualities are exact symmetries of these theories). This adds credibility to the use of Weyl-group symmetries for the groups E_9, E_{10} and E_{11}, where the theory with the full symmetry has not yet been established.

What is *not* demonstrated by this argument is how the space-time signature is encoded in the algebra. For this we do explicitly rely on the non-linear realizations. However, noting that from E_n with $n \geq 3$ on, our theory runs exactly parallel with established results, and using that the extension of the Coxeter group to lower dimensions involves nothing but duality transformations that perform coordinate permutations, we may argue that combining the previous arguments with (Lorentz-)covariance forces the realization of the space-time signature we have established on us.

3.2 Explicit realizations of the root space of E_n

Consider the space defined by n-tuples forming vectors $p = (p_1, \ldots p_n) \in \mathbb{R}^n$, with norm

$$||p||^2 = \sum_{i=1}^{n} p_i^2 - \frac{1}{9}\left(\sum_{i=1}^{n} p_i\right)^2 , \tag{3.14}$$

and inner product $\langle \, , \rangle$ defined by the norm via

$$\langle a, b\rangle = \frac{1}{2}\left(||a+b||^2 - ||a||^2 - ||b||^2\right) = \sum_{i=1}^{n} a_i b_i - \frac{1}{9}\left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} b_i\right) . \tag{3.15}$$

With this inner product, the space always has $n-1$ euclidean directions, as the $n-1$ dimensional subspace of vectors x defined by $\sum_{i=1}^{n} x_i = 0$ contains only vectors of positive norm. The 1 dimensional orthogonal complement consisting of vectors x of the form $x_i = \lambda, \lambda \in \mathbb{R}, \forall i$. These have norm

$$\lambda^2 n \left(1 - \frac{n}{9}\right) ; \tag{3.16}$$

This is positive for $n < 9$, negative for $n > 9$, and zero for $n = 9$. Consequently, these spaces have the right properties to serve as root spaces for the E_n algebra's (these choices were inspired by the choice of metric on the E_{10} root space in [29, 30]).

We realize the simple roots of E_n in the above space as

$$\alpha_{n-i} = e_i - e_{i+1} , \qquad i = 1, \ldots, n-1 ; \tag{3.17}$$

$$\alpha_0 = e_{n-2} + e_{n-1} + e_n . \tag{3.18}$$

Here $(e_i)_j = \delta_{ij}$, and the reader should notice that with the inner product (3.15) these are *not* unit vectors. The root lattice of E_n consists of linear combinations of the simple roots, with coefficients in $\mathbb{Z}$. This lattice can be characterized as consisting of those vectors a whose components a_i are integers, and sum up to three-folds $3k$. The integer k counts the occurrences of the "exceptional" root α_0, and for roots equals the level as defined in [31, 28, 15].

The roots at level 0 coincide with the roots of the algebra A_{n-1}. Its real form is $\mathrm{SL}(n, \mathbb{R})$ whose discrete subgroup $\mathrm{SL}(n, \mathbb{Z})$ is argued to be the symmetry acting on the compactification torus T_n. The intersection of A_{n-1} with H_n provides us with a real form of $so(n, \mathbb{C})$ specifying the space-time signature of the torus T_n.

In the basis we have chosen the Weyl reflections w_i in α_i, $i = 1, \ldots, n$ permute entries of the row of n numbers. The action of these is easily taken into account. Signature changing dualities (not acting as space-time coordinate permutations) are given by Weyl reflections in α_0, and other roots that have α_0 exactly once in their expansion. These are all of the form

$$\beta_{ijk} = e_i + e_j + e_k , \qquad i < j < k .$$

These conventions are convenient for exploring compactifications of M-theories. The fundamental coweights can be easily computed, except for E_9 where the computation is impossible because of the null-direction in the root space. This can however be easily circumvented by embedding the root space of E_9 in that of E_{10}, in the obvious way. The coweights are then specified up to an arbitrary entry expressing the value on the root of E_{10} that is absent in E_9.

To relate to IIA-theories, the root that represents the mixing of the 11^{th} dimension with the rest of space-time obtains a different interpretation. For convenience, we can take it to be node 1. Now the global symmetry on the space-time torus is encoded by the roots $\alpha_2, \ldots, \alpha_n$.

Unfortunately, these conventions are well-adapted to M- and IIA-theory, and therefore slightly inconvenient for IIB-theory. Roughly an analogous choice for IIB-theory would be to use a space $\mathbb{R}^n$ with inner product

$$\langle a, b\rangle = \sum_{i=1}^{n-1} a_i b_i - \frac{1}{8}\left(\sum_{i=1}^{n-1} a_i\right)\left(\sum_{i=1}^{n-1} b_i\right) + \frac{1}{2} a_n b_n\,. \tag{3.19}$$

Again it is easily shown that this inner product is positive definite for $n < 9$, that it has a single null direction for $n = 9$ and has lorentzian signature for $n > 9$.

The simple roots of E_n are then realized as

$$\alpha_{n-i} = e_i - e_{i+1}\,, \qquad i = 1, \ldots, n-3\,; \tag{3.20}$$

$$\alpha_0 = e_{n-2} - e_{n-1}\,; \tag{3.21}$$

$$\alpha_1 = -2e_n\,; \tag{3.22}$$

$$\alpha_2 = e_{n-2} + e_{n-1} + e_n\,. \tag{3.23}$$

We now identify an $\mathrm{SL}(n-1,\mathbb{R})$ group, with simple roots $\alpha_0, \alpha_3, \ldots \alpha_{n-1}$. In these conventions the Weyl reflections $w_0, w_3 \ldots, w_{n-1}$ generate the permutation group on space-time. The Weyl reflection w_1 generates IIB S-duality. Signature changing dualities must come from the Weyl reflection w_2. The root α_2 can be suggestively decomposed as $e_{n-2} + e_{n-1}$ (representing a 2-form) and e_n, signifying that this 2-form is part of a doublet under the $\mathrm{SL}(2,\mathbb{R})$ factor.

These conventions are "nice" for computations that stay well inside IIB. For computations comparing M- and IIA- on the one hand, and IIB on the other hand, the results must either be translated to the other conventions, or one must stick to the abstract formulation.

3.3 Space-time signature; Diagrammatics

The cases of 11 dimensional supergravity, and the 10 dimensional IIA- and IIB-supergravities rely on different $\mathrm{SL}(k,\mathbb{R})$ subalgebra's. It is straightforward to extract the space-time signature from these algebra's following an adaptation of methods of [24, 32, 18]. Essentially, one can count the number of generators of $\mathrm{SL}(k,\mathbb{R})$ of the form (2.10) or (2.19), and divide them into compact and non-compact generators (specified by f). This allows an easy determination of the signature of the $\mathrm{SO}(k-p,p)$ subgroup of $\mathrm{SL}(k,\mathbb{R})$, and hence fixes p.

The remaining generators of E_n are related to (reductions of) form fields. In [18] we argued that for M-theories, the presence of ω_0 in the function f encodes the sign in front of the 4-form term in the 11 dimensional lagrangian. In the IIA interpretation, it can be seen that the root α_0 no longer corresponds to the 3-form potential, but to the 2-form potential that results from dimensional reduction, with its 3-form field strength. The reasoning of [18] carries over the sign of the 3-form field strength term in the IIA lagrangian. The IIA interpretation also alters the interpretation of the root α_1. In the M-theory picture, α_1 is a root of the $\mathrm{SL}(n, \mathbb{R})$ group. In the IIA-picture, it is seen to correspond to the Kaluza-Klein vector. The presence of the coweight ω_1 in the function f encodes the sign in front of the 2-form field strength term in the IIA-lagrangian. From the algebra one can easily see that this sign in unconventional, if and only if the IIA interpretation is related to an 11-dimensional interpretation by time-like reduction. The algebra reproduces the same relation that is well-known from the explicit dimensional reduction procedure.

For the IIB-interpretation we have the roots α_1 and α_2 that are not participating in the space-time symmetry group $\mathrm{SL}(k, \mathbb{R})$. The easiest to interpret of these two is α_1. Being completely orthogonal to the roots of $\mathrm{SL}(k, \mathbb{R})$, it should be clear that this is a root of an $\mathrm{SL}(2, \mathbb{R})$, precisely the $\mathrm{SL}(2, \mathbb{R})$ appearing as a global symmetry in IIB supergravity. The appearance of the coweight ω_1 in the function f specifying the real form indicates whether the denominator subgroup appearing under $\mathrm{SL}(2, \mathbb{R})$ is compact of non-compact, that is whether the 10-dimensional coset is $\mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(2)$ or $\mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(1,1)$. The root α_2 corresponds to two-forms of $\mathrm{SL}(k, \mathbb{R})$, forming a doublet under $\mathrm{SL}(2, \mathbb{R})$. Consequently, the coweight ω_2 must encode information on the sign of the 3-form terms in the IIB-lagrangian. There is however a slight subtlety: If the 10-dimensional coset is $\mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(2)$ both the 3-form terms in the lagrangian have the same sign, and which sign is determined by ω_2. If the coset is $\mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(1,1)$ however, the two 3-forms have different signs. Now ω_2 also contributes to the sign, but in this case its contribution can always be undone by a field redefinition, or alternatively, an S-duality transformation.

There are of course more fields in the IIA and IIB lagrangians. As we will explain in the sections 6 and 7, the signs of these are completely determined in terms of the other signs.

A nice way to visualize the action of the function f is to inscribe its values on the simple roots on the corresponding nodes of the Dynkin diagram. For example for $E_5 \cong D_5$ a particular function f would be encoded by

$$\begin{array}{cccc} & & 1 & \\ 1 & 1 & 0 & 1\,. \end{array}$$

A nice feature of such a visualization is that the action of a fundamental Weyl reflection is easily transcribed onto the diagram. From

$$\langle \alpha_j, w_i(f) \rangle = \langle w_i(\alpha_j), f \rangle = \langle \alpha_j, f \rangle - a_{ij} \langle \alpha_i, f \rangle\,, \tag{3.24}$$

where a_{ij} corresponds to an entry from the Cartan matrix A_n, we deduce that the action of the Weyl reflection w_i on the diagram is described by "add the value of node i to all nodes that are connected to it, and reduce modulo 2".[1]

[1] This prescription applies to all simply laced algebra's. The extension to non-simply laced algebra's is easily deduced from equation (3.24).

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As an example, for $E_4 \cong A_4$, one particular orbit of functions, defined by successive Weyl reflections (which are duality transformations) is described by the diagrams

$$\begin{matrix} & & 0 \\ 1 & 0 & 0 \end{matrix} \leftrightarrow \begin{matrix} & & 0 \\ 1 & 1 & 0 \end{matrix} \leftrightarrow \begin{matrix} & & 0 \\ 0 & 1 & 1 \end{matrix} \leftrightarrow \begin{matrix} & & 1 \\ 0 & 0 & 1 \end{matrix} \leftrightarrow \begin{matrix} & & 1 \\ 0 & 0 & 0 \end{matrix}. \tag{3.25}$$

All these functions describe the coset $\mathrm{SL}(5,\mathbb{R})/\,\mathrm{SO}(4,1)$. The interpretations of the various diagrams differ however. Interpreting the diagrams as referring to $M-$theory, the horizontal line represents the $4-$torus. The first diagram (as well as the following 3) is easily seen to represent a torus signature (t,s) with $|t-s|=2$, while the last diagram corresponds to $|t-s|=4$.

Alternatively, interpreting these diagrams as relevant to IIB, the vertical line corresponds to the $3-$torus, whereas the node at the far end indicates whether we are dealing with the coset $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{SO}(2)$ (if its inscribed value is 0), or $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{SO}(1,1)$ if its inscribed value is 1. We then have the choice between $|t-s|=3$ and $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{SO}(1,1)$, or $|t-s|=1$ and $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{SO}(2)$.

Note how both examples nicely illustrate how, by Weyl reflections/duality transformations, we can have signs from the space-time signature "running" into the gauge sector, and vice versa.

For n sufficiently large the number of diagrams is large. The above procedure is however easily implemented in a simple computer program. We have used such a program to verify the tables we will give in the following sections.

Another easy check on our results is to compare with [18]. There the full classification was done for E_{11}, which is as far as our computation will go, and all functions f relevant to compactifications of M-theory were found.

By erasing a suitable node of the Dynkin diagram of E_{11}, the diagram splits into the two diagrams of E_n and A_{10-n}. Inscribing the values of the possible functions f on the nodes, one easily finds the signature of the transverse space-time from A_{10-n}, whereas the real form of the denominator subgroup and the torus signature follow from the values of f on E_n.

4. The duality web for M-theories: Finite dimensional groups

We are primarily interested in the denominator subgroups that can appear. In 11 and 10 dimensions, we have the variants of M- and IIA-theory which were already completely described in [4] (see also [17]). We therefore proceed immediately to 9 dimensions, which is the first time a semi-simple factor appears in the algebra.

4.1 Coset symmetries in 9 dimensions

In 9 dimensions the global symmetry group has algebra $gl(2,\mathbb{R}) \cong sl(2,\mathbb{R}) \oplus \mathbb{R}$. This algebra has a simple geometric interpretation, the $\mathbb{R}$ being related to rescaling the volume of the two dimensions that were reduced away (the volume of the 2-torus if we are discussing toroidal compactification), and the $sl(2,\mathbb{R})$-term related to volume preserving transformations.

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The simple factor is generated by $sl(2,\mathbb{R})$, which is a real form of A_1. The algebra A_1 has only two real forms, and no outer automorphisms, so both are suitable to our construction. The denominator subgroup is a 1 dimensional abelian group, which we denote by T_1.[2] The real forms, and the predicted abelian subgroups are

$$\begin{array}{rlcll} su(2): & \sigma(A_1) = -3 & \rightarrow & \sigma(T_1) = -1 & : so(2)\,; \\ sl(2,\mathbb{R}): & \sigma(A_1) = 1 & \rightarrow & \sigma(T_1) = 1 & : so(1,1). \end{array} \tag{4.1}$$

Our group theory predicts that there are actually two possible real forms for the denominator algebra, with $\sigma = \pm 1$. The denominator subgroup is an abelian group, $\sigma = 1$ indicates it must be the compact $so(2) \cong u(1)$, while $\sigma = -1$ reveals the non-compact $so(1,1)$.

Of course $so(2)$ occurs when reducing over two space-, or two time-like directions respectively, and $so(1,1)$ when there is one space- and one time-like direction included [17]. We have only included these here to demonstrate that our techniques also work fine with the simplest examples.

For the relation to IIB-theories, we refer the reader to section 7.

4.2 Coset symmetries in 8 dimensions

In 8 dimensions, the global symmetry group has algebra $sl(3,\mathbb{R}) \oplus sl(2,\mathbb{R})$. From 8 and less dimensions the global symmetry algebra's are semi-simple.

We have discussed the real forms for $sl(2,\mathbb{R})$ in the previous subsection. The algebra $sl(3,\mathbb{R})$ is a real form of A_2, which has 3 real forms, known as $sl(3,\mathbb{R})$, $su(2,1)$ and $su(3)$. The denominator subalgebra must be a real form of A_1. But as $sl(3,\mathbb{R})$ is generated by a Cartan involution that is outer, it does not match our criteria (notice that its $\sigma(A_2) = 3$ would have resulted in $\sigma(A_1) = 2$ which is impossible for A_1). The remaining real forms imply the following possibilities for the denominator sub-algebra:

$$\begin{array}{rlcll} su(3): & \sigma(A_2) = -8 & \rightarrow & \sigma(A_1) = -3 & : so(3)\,; \\ su(2,1): & \sigma(A_2) = 0 & \rightarrow & \sigma(A_1) = 1 & : so(2,1)\,. \end{array} \tag{4.2}$$

Combining these with the results of the previous section, one finds that there are 4 possibilities for the denominator sub-algebra, being $so(3) \oplus so(2)$, $so(3) \oplus so(1,1)$, $so(2,1) \oplus so(2)$ and $so(2,1) \oplus so(1,1)$.

The $\mathrm{SL}(2,\mathbb{R})$ factor appears because the (axionic) scalar ψ, formed from reducing the 11-dimensional 3 form over the 3-torus, combines with the (dilatonic) scalar ϕ representing the volume of this 3-torus, into a realization of the coset $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{SO}(2)$ or $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{SO}(1,1)$. The difference between the two cosets manifests itself as a relative sign between the dilatonic and the axionic scalar. The relevant part of the lagrangian is

$$-\frac{1}{2}\left(*\mathrm{d}\phi \wedge \mathrm{d}\phi \pm e^{2\phi} *\mathrm{d}\psi \wedge \mathrm{d}\psi \right), \tag{4.3}$$

The plus appears for the coset $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{SO}(2)$, while the minus-sign indicates the coset $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{SO}(1,1)$. Which one of the two is realized depends on two signs: The sign of the

[2] This notation reflects that an abelian group is called a torus. Unlike the usual meaning of the word in physics, the mathematical notion does not require the group to be compact.

$\mathbb{R}_{m,n}$	$T_{p,q}$				h
$(0,8)$	$(1,2)$	$(2,1)$			$so(2,1) \oplus so(1,1)$
$(1,7)$	$(0,3)$				$so(3) \oplus so(2)$
$(1,7)$		$(1,2)$			$so(2,1) \oplus so(2)$
$(2,6)$		$(0,3)$	$(3,0)$		$so(3) \oplus so(1,1)$
$(3,5)$			$(2,1)$		$so(2,1) \oplus so(2)$
$(3,5)$				$(3,0)$	$so(3) \oplus so(2)$
$(4,4)$			$(1,2)$	$(2,1)$	$so(2,1) \oplus so(1,1)$

Table 1: Dualities of M-theories in 8 dimensions.

4-form term in the 11-dimensional theory, and a sign coming from the signature of the 3 dimensions one is reducing over.

The theories in signatures $(1,10)$ $(6,5)$, and $(9,2)$ have a conventional sign in front of the 4-form kinetic term [4], giving a positive sign in our computation, whereas the other ones have an extra minus sign. The sign coming from the 3-torus is plus if the number of time-dimensions is even; otherwise it is minus.

The sign in equation (4.3) is given by multiplying these two signs. The plus, indicating SO(2) as denominator subgroup, appears if the 11-dimensional 4-form term is conventional and the 3 dimensions include an even number of time directions, or if there is an unconventional sign in 11-dimensions, combined with an odd number of time directions. In the other cases, the minus sign appears, and the denominator subgroup is $\mathrm{SO}(1,1)$.

We summarize our findings in table 1.

Table 1 is the first of a series of tables that all exhibit the same structure. In the column under $\mathbb{R}_{m,n}$ we have denoted the signature of the transverse space-time. Then there are a number of columns representing tori $T_{p,q}$ of various signatures. The reader can reconstruct which theory we are dealing with by simply computing the signature $(m+p, n+q)$ of the overall space-time. Each row in our table represents a group of theories that can be transformed into one another by duality symmetries. The last entry in the row is the denominator sub-algebra h, that is common to all entries in the row. The various descriptions in each row are related by dualities. The space-time signatures that are not in this table have $n < m$ in $\mathbb{R}_{m,n}$. The answer for these theories can be found by simply interchanging $m \leftrightarrow n$, $p \leftrightarrow q$, which gives all the remaining configurations.

The groups $\mathrm{SO}(3) \times \mathrm{SO}(2)$, $\mathrm{SO}(2,1) \times \mathrm{SO}(1,1)$ and $\mathrm{SO}(2,1) \times \mathrm{SO}(2)$ can be found in [16, 17, 3, 33, 34].[3] In addition we find that also the group $\mathrm{SO}(3) \times \mathrm{SO}(1,1)$ is possible, and that it is crucial to complete the duality web.

When compactifying on a 3-torus, one can reach at most one of the other theories. This is always accompanied with a double sign change: Both the four-form term, and the signature of the torus are different. That in all those cases the theories are described by a duality group that has as second factor $\mathrm{SO}(1,1)$ is a coincidence; dropping the requirement

[3] The papers [16, 3] contain an unfortunate typo in their answer for time-like compactification of conventional 11 dimensional supergravity to 8 dimensions.

of supersymmetry, and studying other signatures one sees that SO(2) is not ruled out by principle, but simply does not occur in the M-theory duality chain.

4.3 Coset symmetries in 7 dimensions

For 7 dimensions and below, the space-time symmetries and the 3-form gauge field merge into a simple global symmetry group. In 7 dimensions the relevant algebra is $sl(5,\mathbb{R}) \cong A_4$. A_4 has 4 inequivalent real forms, of which 3 are generated by inner automorphisms. The denominator sub-algebra must be a real form of B_2, and doing the computation leads to the following possibilities:

$$\begin{aligned} so(5): &\quad \sigma(A_4) = -24 &\quad \rightarrow &\quad \sigma(B_2) = -10 &: so(5)\,; \\ su(4,1): &\quad \sigma(A_4) = -8 &\quad \rightarrow &\quad \sigma(B_2) = -2 &: so(4,1)\,; \\ su(3,2): &\quad \sigma(A_4) = 0 &\quad \rightarrow &\quad \sigma(B_2) = 2 &: so(3,2)\,. \end{aligned} \tag{4.4}$$

This computation has reproduced the groups found in [16, 17, 3].

The symmetries of space-time are embedded in a real form of the algebra $so(4,\mathbb{C}) \cong A_1 \oplus A_1$. To obtain the space-time signature, one has to study the embedding of A_3 in A_4, and the consequences for the $A_1 \oplus A_1$ embedding in B_2. Even without the details, it is already clear that $so(5) \supset so(4)$, that $so(4,1) \supset so(4), so(3,1)$ and $so(3,2) \supset so(3,1), so(2,2)$. The generators that are in B_2, but not in $A_1 \oplus A_1$ correspond to the 4-form sector, and there is a sign difference for the corresponding scalars if for example the $so(4)$ comes from $so(4,1)$ or $so(5)$. Tracing this sign back to 11 dimensions we arrive at table 2.

Obviously, the compactification of $M_{(1,10)}$ theory on a euclidean 4-torus stands isolated, and corresponds to the symmetry algebra $so(5)$. By symmetry, the same applies to $M_{(10,1)}$ on a time-like 4-torus. Also for the theory in space-time signature $(6,5)$, compactified on a torus with spatial directions only the duality algebra is $so(5)$ and a transition to another theory is not possible. This is due to the fact that the $(6,5)$ theory has the conventional four-form sign. By symmetry, the same applies to $(5,6)$-theory on a time-like 4-torus.

For all other combinations of signs, duality transitions between theories of different signatures are possible.

$\mathbb{R}_{m,n}$	$T_{p,q}$				h
$(0,7)$	$(1,3)$	$(2,2)$			$so(3,2)$
$(1,6)$	$(0,4)$				$so(5)$
$(1,6)$		$(1,3)$	$(4,0)$		$so(4,1)$
$(2,5)$		$(0,4)$	$(3,1)$		$so(4,1)$
$(2,5)$				$(4,0)$	$so(5)$
$(3,4)$			$(2,2)$	$(3,1)$	$so(3,2)$

Table 2: Dualities of M-theories in 7 dimensions.

$\mathbb{R}_{m,n}$	$T_{p,q}$				h
$(0,6)$	$(1,4)$	$(2,3)$	$(5,0)$		$so(5,\mathbb{C})$
$(1,5)$	$(0,5)$				$so(5) \oplus so(5)$
$(1,5)$		$(1,4)$	$(4,1)$		$so(4,1) \oplus so(4,1)$
$(1,5)$				$(5,0)$	$so(5) \oplus so(5)$
$(2,4)$		$(0,5)$	$(3,2)$	$(4,1)$	$so(5,\mathbb{C})$
$(3,3)$			$(2,3)$	$(3,2)$	$so(3,2) \oplus so(3,2)$

Table 3: Dualities of M-theories in 6 dimensions.

4.4 Coset symmetries in 6 dimensions

In 6 dimensions, the global symmetry is Spin(5,5). The denominator subgroups that can occur are given by our standard computation based on equation (2.28):

$$
\begin{array}{llcll}
so(10): & \sigma(D_5) = -45 & \to & \sigma(B_2 \oplus B_2) = -20 & : so(5) \oplus so(5)\,; \\
so(2,8): & \sigma(D_5) = -13 & \to & \sigma(B_2 \oplus B_2) = -4 & : so(4,1) \oplus so(4,1)\,; \\
so^*(10): & \sigma(D_5) = -5 & \to & \sigma(B_2 \oplus B_2) = 0 & : so(5,\mathbb{C})\,; \\
so(4,6): & \sigma(D_5) = 3 & \to & \sigma(B_2 \oplus B_2) = 4 & : so(3,2) \oplus so(3,2)\,.
\end{array}
\tag{4.5}
$$

The computation of the signature is not decisive in the case when the signature $\sigma(B_2 \oplus B_2) = 0$, at first sight this leaves $so(4,1) \oplus so(3,2)$ and $so(5,\mathbb{C})$ as options. Studying the embedding of the $B_2 \oplus B_2$ algebra in D_5, however, it is easily seen that there is a symmetry between the two B_2-factors, ruling out the first option and fixing the algebra to be the one of $so(5,\mathbb{C})$.

Although they can be computed directly, also here a fairly intuitive way of understanding the possible space-time signatures is possible. Studying how the rotations in the 5 reduced dimensions are embedded in the $so(5) \oplus so(5)$ of the $(1,10)$ theory reduced on 5 euclidean dimensions, one sees that it is the diagonal algebra that represents the symmetries of these dimensions. Correspondingly, the diagonal algebra in $so(4,1) \oplus so(4,1)$ is obviously $so(4,1)$, and the one in $so(3,2) \oplus so(3,2)$ is $so(3,2)$, so these theories must correspond to torus signatures $(4,1)$ and $(1,4)$, and $(3,2)$ and $(2,3)$ respectively.

For the $so(5,\mathbb{C})$ algebra the situation is a little more involved. The embeddings follow from writing the algebra as $so(5) \oplus i\ so(5)$, where i is the imaginary unit. But as also $so(3,2) \oplus i\ so(3,2)$ and $so(4,1) \oplus i\ so(4,1)$ result in $so(5,\mathbb{C})$, it appears that all signatures are possible, and this is indeed confirmed in a direct computation. Putting all results together, we arrive at table 3.

Notice that in this dimension, for the first time, there is a duality group that connects 3 M-theories. It seems that the algebra $so(3,2) \oplus so(3,2)$ has not appeared in the literature before. The reader who studies our table 3 will notice that it occurs in a part of the duality web not studied in detail in [3, 4], only occurring for the $(6,5)$ and $(5,6)$ theory compactified on a torus of signature $(3,2)$ or $(2,3)$. There is therefore no contradiction with earlier results, though the algebra $so(3,2) \oplus so(3,2)$ is really necessary to complete the duality web.

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$\mathbb{R}_{m,n}$	$T_{p,q}$				h
$(0,5)$	$(1,5)$	$(2,4)$	$(5,1)$		$sp(2,2)$
$(0,5)$				$(6,0)$	$sp(4)$
$(1,4)$	$(0,6)$				$sp(4)$
$(1,4)$		$(1,5)$	$(4,2)$	$(5,1)$	$sp(2,2)$
$(2,3)$		$(0,6)$	$(3,3)$	$(4,2)$	$sp(4,\mathbb{R})$

Table 4: Dualities of M-theories in 5 dimensions.

4.5 Coset symmetries in 5 dimensions

In 5 dimensions, the global symmetry becomes the exceptional $E_{6(6)}$. Our usual computation for the possible denominator subgroups gives:

$$\begin{aligned}
e_{6(-78)}: &\quad \sigma(E_6) = -78 &\rightarrow&\quad \sigma(C_4) = -36 &: sp(4)\,;\\
e_{6(-14)}: &\quad \sigma(E_6) = -14 &\rightarrow&\quad \sigma(C_4) = -4 &: sp(2,2)\,;\\
e_{6(2)}: &\quad \sigma(E_6) = 2 &\rightarrow&\quad \sigma(C_4) = 4 &: sp(4,\mathbb{R})\,.
\end{aligned} \tag{4.6}$$

The space-time rotations in the compact group $sp(4)$ are hidden in its regular subalgebra $u(4) \cong su(4) \oplus u(1)$, where one has to remember that $su(4) \cong so(6)$. Similar relations are true for the other real forms of A_3, as $su^*(4) \cong so(1,5)$, $su(2,2) \cong so(4,2)$ while $sl(4,\mathbb{R}) \cong so(3,3)$. It then remains to identify the subalgebra's: $sp(4) \supset su(4)$, $sp(2,2) \supset su^*(4), su(2,2)$ (the first of these embeddings can be easily seen from the Satake diagram of $su^*(4)$), and $sp(4,\mathbb{R}) \supset sl(4,\mathbb{R}), su(2,2)$. The reader less familiar with these groups may want to try an explicit computation.

Apart from the groups mentioned previously in the literature, we also find the group $\mathrm{Sp}(4,\mathbb{R})$. Table 4 reveals its place in the duality web. The group $\mathrm{Sp}(4,\mathbb{R})$ appears in a part of the duality web not explored in detail in [3, 4], so there is no contradiction with earlier results. Note furthermore that again the $(6,5)$ and $(5,6)$ theories allow a compact denominator group for a 6-torus with space- or time-like dimensions only. This is the last dimension for which this happens, for a 7-torus the signature must be mixed for $(6,5)$ and $(5,6)$ theories.

4.6 Coset symmetries in 4 dimensions

In 4 dimensions, the global symmetry of the theory is $E_{7(7)}$. The algebra E_7 has no outer automorphisms, and as a matter of fact none of the E_n algebra's with $n \geq 7$ has. Hence we find a one to one correspondence between possible real forms of $E_{7(7)}$, and the possible real forms of the algebra A_7 that can appear in the denominator sub-algebra. Our computation gives the following possibilities:

$$\begin{aligned}
e_{7(-133)}: &\quad \sigma(E_7) = -133 &\rightarrow&\quad \sigma(A_7) = -63 &: su(8)\,;\\
e_{7(-25)}: &\quad \sigma(E_7) = -25 &\rightarrow&\quad \sigma(A_7) = -9 &: su^*(8)\,;\\
e_{7(-5)}: &\quad \sigma(E_7) = -5 &\rightarrow&\quad \sigma(A_7) = 1 &: su(4,4)\,;\\
e_{7(7)}: &\quad \sigma(E_7) = 7 &\rightarrow&\quad \sigma(A_7) = 7 &: sl(8,\mathbb{R})\,.
\end{aligned} \tag{4.7}$$

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$\mathbb{R}_{m,n}$	$T_{p,q}$					h
$(0,4)$	$(1,6)$	$(2,5)$	$(5,2)$	$(6,1)$		$su^*(8)$
$(1,3)$	$(0,7)$					$su(8)$
$(1,3)$		$(1,6)$	$(4,3)$	$(5,2)$		$su(4,4)$
$(2,2)$		$(0,7)$	$(3,4)$	$(4,3)$	$(7,0)$	$sl(8,\mathbb{R})$

Table 5: Dualities of M-theories in 4 dimensions.

$\mathbb{R}_{m,n}$	$T_{p,q}$					h
$(0,3)$	$(1,7)$	$(2,6)$	$(5,3)$	$(6,2)$		$so^*(16)$
$(1,2)$	$(0,8)$					$so(16)$
$(1,2)$		$(1,7)$	$(4,4)$	$(5,3)$	$(8,0)$	$so(8,8)$

Table 6: Dualities of M-theories in 3 dimensions.

These organize in the duality web according to table 5.

We have again found a possible denominator group, $\mathrm{SL}(8,\mathbb{R})$, that has not appeared in the literature before, but again there is no contradiction with earlier results.

4.7 Coset symmetries in 3 dimensions

In 3 dimensions the global symmetry group is $E_{8(8)}$. The algebra E_8 can appear in 3 real forms, which lead to the following possibilities for denominator subgroups.

$$\begin{aligned} e_{8(-248)}: &\quad \sigma(E_8) = -248 &&\rightarrow &&\sigma(D_8) = -120 &&: so(16)\,; \\ e_{8(-24)}: &\quad \sigma(E_8) = -24 &&\rightarrow &&\sigma(D_8) = -8 &&: so^*(16)\,; \\ e_{8(8)}: &\quad \sigma(E_8) = -8 &&\rightarrow &&\sigma(D_8) = 8 &&: so(8,8)\,. \end{aligned} \tag{4.8}$$

The duality web is reproduced in table 6.

This time we only find groups encountered previously in the literature. All M-theories and all possible signatures for the 8-torus are described by this small set of groups.

5. Low dimensions: Infinite dimensional groups

Reducing to less than 3 dimensions, we encounter infinite dimensional groups. For the application we are discussing here, it does not really matter whether we are discussing dimensionally reduced theories [9], or conjectured formulations of the full, unreduced theory with a hidden symmetry (such as the proposals of [15, 19]) , essentially because the cosets refer to the zero-mode spectrum only. As the real forms of the infinite groups we encounter are unfamiliar, we devote some discussion to explicit realizations.

5.1 2 dimensions: real forms of E_9 and H_9

In two dimensions we are dealing with the coset $E_{9(9)}/H_9$. There are several new features, the most significant one of course being the fact that we are dealing with infinite-dimensional groups here. Furthermore we will find a group that according to our previous criteria can appear as a denominator subgroup, but detailed computation reveals that it

$\mathbb{R}_{m,n}$	$T_{p,q}$					h
$(0,2)$	$(1,8)$	$(2,7)$	$(5,4)$	$(6,3)$	$(9,0)$	H_9^{n1}
$(1,1)$	$(0,9)$					H_9^c
$(1,1)$		$(1,8)$	$(4,5)$	$(5,4)$	$(8,1)$	H_9^{n2}

Table 7: Dualities of M-theories in 2 dimensions.

does not appear in the M-theory duality web, because it can only correspond to space-time signatures incompatible with supersymmetry.

The denominator subgroup H_9 can appear in 4 possible real forms, that we will denote by $H_9^c, H_9^{n1}, H_9^{n2}$ and H_9^{n3}. The group H_9^c is the compact real form, and has been denoted in the past as $K(E_9)$ and $\mathrm{SO}(16)^\infty$ [8, 12, 30]. The groups H_9^{n1}, H_9^{n2} and H_9^{n3} are non-compact real forms. We should not by analogy to the compact case denote these by $\mathrm{SO}(8,8)^\infty$ or $SO^*(16)^\infty$ or similar; such a notation would be ambiguous, as for example $SO^*(16)$ is a subgroup of both H_9^{n1} and H_9^{n2}, and on the other hand H_9^{n2} has $\mathrm{SO}(8,8)$ as well as $SO^*(16)$ as subgroups. The reader can verify this by direct computation, or by using our tables and decompactifying time and space-like dimensions in the theories with these groups. Which real form of $H_9^{\mathbb{C}}$ is obtained depends on how one constructs the group from the infinite tower of D_8 representations. Note that the notion of signature of the algebra, which distinguishes the finite dimensional real forms is ill-defined for these infinite dimensional groups

Only the groups H_9^c, H_9^{n1} and H_9^{n2} appear in the duality web, that is reflected in table 7.

The fourth real form H_9^{n3} does not appear in this table. It is associated to combinations of space-time signatures, and signs for the gauge fields that cannot appear in compactifications of theories that descend from an 11 dimensional supersymmetric theory.

A particular way to construct H_9^{n1} is as follows: First we make an E_9 level decomposition, by decomposing with respect to the horizontal $A_8 = \mathrm{SL}(9,\mathbb{R})$ algebra. When we project to H_9, we turn all generators that are composed of ladder operators at levels $\pm k$ where k is odd, into non-compact ones, and the remaining ones into compact ones. It is easily seen that the algebra obtained this way closes.

For H_9^{n2} we make the E_9 level decomposition that is more common in the mathematical literature, with respect to E_8, and then decompose to $\mathrm{SO}(16)$. We have an infinite tower of repeating irreps, that are either the **120** or the spin irrep **128**. Projecting to H_9 all **120** irreps are paired, except for the one at level 0. We pair the **128** at level k with the one at level $-k$, the **128** at level 0 is paired with itself. The **120** irreps are then projected to compact generators while all the **128** correspond to non-compact ones. This is the real form of H_9^{n2} that corresponds to the split real form of $E_{9(9)}$.

For H_9^{n3} we make an E_9 level decomposition in E_8 irreps, and then decompose these to $E_7 \oplus su(2)$. Under this decomposition $\mathbf{248} \to (\mathbf{133},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3}) \oplus (\mathbf{56},\mathbf{2})$. When projecting to H_9 we turn all generators of $(\mathbf{56},\mathbf{2})$ into non-compact ones, and the remaining ones to compact ones. The corresponding real form of E_9 can be characterized as the affine Lie algebra built on the real form $E_{8(-24)}$ of E_8.

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$\mathbb{R}_{m,n}$	$T_{p,q}$						h
$(1,0)$		$(1,9)$	$(4,6)$	$(5,5)$	$(8,2)$	$(9,1)$	H_9^{n1}
$(1,0)$	$(0,10)$						H_9^c

Table 8: Dualities of M-theories in 1 dimensions.

The coset $E_{8(-24)}/(E_7 \times \mathrm{SU}(2))$ can appear in a 3 dimensional coset theory [35]. This theory allows a supersymmetric extension (with at most 8 supersymmetries), and can in turn be oxidized to a 6 dimensional theory [22]. It should be expected that the real form of E_9 implied by H_9^{n3} will appear in the compactification of this theory to 2 dimensions.

5.2 1 dimension: real forms of E_{10} and H_{10}

For $E_{10(10)}$ there are three possible real forms. In the theories that descend from an 11 dimensional supersymmetric theory, only the compact form, and one of the two real forms can occur.

We have summarized the results of the computation in table 8, although it has almost trivial content.

The compact real form H_{10}^c represents the symmetries of the theories in space-time signatures $(1,10)$ and $(10,1)$, where the single time, resp. space dimension is kept transverse, and the other ones are compactified. The non-compact form we have called H_{10}^{n1} describes all the other situations arising from compactification of 11 dimensional supergravity theories. Another non-compact form H_{10}^{n2} describes theories in space-time signatures, or with signs in front of the 4-form gauge-field terms that cannot occur in the M-theory duality web.

A particular way to construct H_{10}^{n1} is as follows: Decompose E_{10} with respect to the leftmost node into $D_9 \cong so(18)$ irreps. The result is an infinite towers of irreps of which some are congruent to either of the two spin-irreps,[4] and some are congruent to the vector or adjoint irreps. We then project to H_{10}, by setting all generators that correspond to irreps congruent to spin irreps of $so(18)$ to non-compact generators, and the rest to compact ones.

The real form H_{10}^{n2}, that cannot occur in the M-theory duality web can be constructed as follows. We decompose with respect to the exceptional node, into SL(10) irreps. We then project all generators at odd levels (where now we define level with respect to SL(10)) to non-compact generators, and all the generators at even levels to compact ones.

5.3 0 dimensions: real forms of E_{11} and H_{11}

The essential computations for this case have been done in [18], where they where elaborated upon in great detail. We therefore only repeat the conclusions of this paper: There are 4 possible denominator subgroups. Of these, only one corresponds to the signs appropriate for M-theory and its cousins, the M^* and M' theories. Moreover, it can be demonstrated that this choice of signs allows all the 11 dimensional supergravity theories, but no others.

[4]For the reader unfamiliar with the concept of congruent irreps: this means that the weights of these irreps correspond to a weight of a spin irrep, plus an element of the root lattice.

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It may be worth noting that, with $n \geq 3$, for E_n with n odd we have always found 4 possible real forms, whereas for E_n with n there appear to be always 3 real forms (are restricting to real forms generated by inner involutions). In spite of the empirical truth of this assertion (we have checked it also for some cases of E_n with $n > 11$), we have not managed to find a simple proof of it.

6. The duality web for IIA theories

All variant IIA-theories can be found by suitable compactification of M-theories on either a time- or a space-like circle. The relations are [17, 3, 4]

$$\begin{array}{ccc} M_{(1,10)} & M_{(2,9)} & M_{(5,6)} \\ s \swarrow \searrow t & s \swarrow \searrow t & s \swarrow \searrow t \\ IIA_{(1,9)}\ IIA_{(0,10)} & IIA_{(2,8)}\ IIA^*_{(1,9)} & IIA_{(5,5)}\ IIA_{(4,6)} \end{array} \tag{6.1}$$

$$\begin{array}{ccc} M_{(10,1)} & M_{(9,2)} & M_{(6,5)} \\ s \swarrow \searrow t & s \swarrow \searrow t & s \swarrow \searrow t \\ IIA_{(10,0)}\ IIA_{(9,1)} & IIA^*_{(9,1)}\ IIA_{(8,2)} & IIA_{(6,4)}\ IIA^*_{(5,5)} \end{array} \tag{6.2}$$

Here s signifies compactification on a space-like circle, whereas t stands for compactification on a time-like circle. It follows that the duality groups for the resulting variants of IIA-theory can be immediately deduced from their M-theory ancestors, as a IIA-theory in space-time signature (p, q) has the same dualities as the M-theory in signature $(p+1, q)$ if the theories are related by compactification on a time-like circle, whereas it has the same dualities as M-theory in signature $(p, q + 1)$ if they are related by compactification on a space-like circle. In particular, table 3 from [3] is reproduced by collecting from our tables the entries for compactification of $M_{(2,9)}$-theory, onto a torus with 1-time direction, leaving 1 other time-direction for the transverse space.

A useful remark in this context is that the fact that the IIA-theories derive from compactification of M-theories implies that there are relations between the signs of various terms. Alternatively, these signs are reflected in the algebra, and can also be derived from this perspective.

In particular the $B_{(2)}$-field 2-form, and the $C_{(3)}$-form in IIA have the same 11 dimensional origin. The sign of kinetic term of the 10-dimensional $C_{(3)}$-form is inherited from its 11 dimensional ancestor. The $B_{(2)}$-term picks up an extra sign if one reduces over a time-like direction. The sign of the Kaluza-Klein vector term also has an unconventional sign precisely for time-like compactification. Consequently, one always has the identity

$$\mathrm{sign}(C_{(1)}) \cdot \mathrm{sign}(C_{(3)}) = \mathrm{sign}(B_{(2)})\,. \tag{6.3}$$

Comparing with table 1 in [4] this is easily verified. Actually this relation extends beyond supergravity. Having a theory with the same field content as the bosonic sector of IIA theory it can only be oxidized to 11 dimensions, and it will have only have the proper symmetric space structure in lower dimensions if equation (6.3) is obeyed.

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We see that the real form of the IIA theory is completely specified by: the space-time signature; the sign of $C_{(1)}$ denoted as σ_1; and the sign of $C_{(3)}$, denoted as σ_3.

Collecting these in the generalized signature $(t, s, \sigma_1, \sigma_3)$, we note

$$(t, s, \sigma_1, \sigma_3) = (s, t, -\sigma_1, -\sigma_3) \,. \tag{6.4}$$

Note that the generalized signature of IIA carries just as much information as the generalized signature for the 11 dimensional theory (see [18]). The sign of the "missing" dimension is encoded in σ_1.

7. The duality web for IIB-theories

There are two kinds of IIB theories. The first kind has two ten-dimensional scalars parameterizing the coset $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{SO}(2)$. For the second kind the two scalars parameterize the coset $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{SO}(1,1)$. In Hull's notation [3, 4] these are denoted as IIB^* and IIB'. The relevant cosets are captured by the part of the lagrangian exhibited in equation (4.2). Another easy way to distinguish them, from the bosonic perspective, is to look at the two 3-form field strengths of the 10 dimensional theory, that form a doublet under the $\mathrm{SL}(2,\mathbb{R})$ global symmetry. As the two fields also form a doublet under the denominator subgroup, they will appear in the quadratic combination

$$-\frac{1}{2}\left(e^{\phi} *H_{(3)} \wedge H_{(3)} \pm e^{-\phi} *G_{(3)} \wedge G_{(3)}\right) . \tag{7.1}$$

The $+$ sign appears for $\mathrm{SO}(2)$, the minus sign for $\mathrm{SO}(1,1)$. The Weyl reflection in the single positive root of $\mathrm{SL}(2,\mathbb{R})$ sends $\phi \to -\phi$, and $H_{(3)} \leftrightarrow G_{(3)}$, and therefore changes the overall sign in the $\mathrm{SO}(1,1)$ case. This is the difference between the IIB^* and IIB' theories [4]. From the supergravity perspective the distinction between the two three-forms is arbitrary. In particular, compactification of either of the two theories leads to the same low energy theory.

As in the IIA-theory there are relations between the possible signs. We cannot appeal to a higher dimensional origin of various signs, but it is easily demonstrated that the algebra leads to relations.

The sign of the axion $C_{(0)}$ is plus if the coset denominator group is $\mathrm{SO}(2)$ and minus if the group is $\mathrm{SO}(1,1)$. The two 2-form potentials $B_{(2)}$ and $C_{(2)}$ appear as a doublet under the abelian group. The quadratic combination appearing in the action has a relative minus sign between the two components if the group is $\mathrm{SO}(1,1)$ (see equation (7.1)).

The 4-form $C_{(4)}$ arises in the algebra in the commutator of the two 2-forms. Putting all these signs together one should have

$$\mathrm{sign}(C_{(0)}) = \mathrm{sign}(B_{(2)}) \cdot \mathrm{sign}(C_{(2)}) = \mathrm{sign}(C_{(4)}) \,. \tag{7.2}$$

Again, with table 2 from reference [4], this is easily verified. It is therefore sufficient to specify the signs of the two 2-form terms.

We encode everything in a generalized signature $(t, s, \sigma_2, \sigma_2')$, where we denote by σ_2, σ_2' the signs of the 2-form terms respectively. If $\sigma_2 \neq \sigma_2'$ one can interchange the two by an S-duality transformation ($=$ Weyl reflection w_1).

We now have the equality

$$(t, s, \sigma_2, \sigma_2') = (s, t, \sigma_2, \sigma_2') \,. \tag{7.3}$$

There is no sign change in the forms ! This can also be seen by noting that the number of two-forms running over an even number of time-like directions is

$$\binom{t}{0}\binom{s}{2} + \binom{s}{0}\binom{t}{2} \tag{7.4}$$

whereas the number of 2-forms running over an odd number of time directions is

$$\binom{t}{1}\binom{s}{1} \,. \tag{7.5}$$

Both these formula's are invariant under $t \leftrightarrow s$ (Compare with the analogous discussion on the 3-form in [18]).

7.1 Tables for IIB-theories

In this section we will list tables representing groups and dualities for all toroidal compactifications of IIB-theories. The tables will have the by now familiar structure, apart from one new ingredient.

For the 11 dimensional M-theories, we have specified the space-time signature. There is also an adjustable sign for the 4-form term, but within the set of supersymmetric M-theories, this sign is completely determined by the space-time signature. This is not so for the IIB-theories, and hence we have indicated in our tables below also the two signs σ_2, σ_2', above the columns. The connection with Hull's notation [3, 4] is simple: if $\sigma_2 = \sigma_2'$ the theory corresponds to a IIB-theory without prime or star, whereas if $\sigma_2 \neq \sigma_2'$ the theory is IIB'/IIB^*.

As the IIB'- and IIB^*-theories are related by a simple field redefinition (which neither the low-energy theory nor our algebra can distinguish), they will be collected under a single entry.

We will not list the 10 and 9-dimensional theories. The 10 dimensional theories are described in [3, 4]. For the 9 dimensional theories there are some signs for the form terms, depending on whether one chooses to compactify on a space- or a time-like direction. These are straightforward to work out, and the coset symmetry in 9 dimensions is the same as in 10 dimensions [17].

The subalgebra's of $sl(3,\mathbb{R}) \oplus sl(2,\mathbb{R})$, relevant to 8 dimensional theories are collected in 9. In 8 dimensions all IIB-theories living in the same space-time signature, but with different signs for the forms become T-dual under compactification on a 2-torus of suitable signature. Of course this is a consequence of the fact that having 2 directions at our disposal, we can link either one of them to an intermediate IIA-theory.

Note that a transition to another theory is possible if and only if the duality algebra contains a $so(2,1)$-term; this is due to the fact that "adjacent" theories have an $so(1,1)$ resp. $so(2)$ local symmetry, that both have to be contained in the real form of A_1 that is relevant to compactification to 8 dimensions, and hence it must be $so(2,1)$.

$\mathbb{R}_{m,n}$	$T_{p,q}$					h
	$++$	$+-$	$--$	$+-$	$--$	
$(0,8)$	$(1,1)$	$(1,1)$				$so(2,1)\oplus so(1,1)$
$(1,7)$	$(0,2)$					$so(3)\oplus so(2)$
$(1,7)$		$(0,2)$	$(2,0)$			$so(2,1)\oplus so(2)$
$(2,6)$			$(1,1)$			$so(3)\oplus so(1,1)$
$(3,5)$			$(0,2)$	$(2,0)$		$so(2,1)\oplus so(2)$
$(3,5)$					$(0,2)$	$so(3)\oplus so(2)$
$(4,4)$				$(1,1)$	$(1,1)$	$so(2,1)\oplus so(1,1)$

Table 9: Dualities of IIB-theories in 8 dimensions.

$\mathbb{R}_{m,n}$	$T_{p,q}$					h
	$++$	$+-$	$--$	$+-$	$++$	
$(0,7)$	$(1,2)$	$(1,2)$	$(3,0)$			$so(3,2)$
$(1,6)$	$(0,3)$					$so(5)$
$(1,6)$		$(0,3)$	$(2,1)$			$so(4,1)$
$(2,5)$			$(1,2)$	$(3,0)$		$so(4,1)$
$(2,5)$					$(3,0)$	$so(5)$
$(3,4)$			$(0,3)$	$(2,1)$	$(2,1)$	$so(3,2)$

Table 10: Dualities of IIB-theories in 7 dimensions.

The reader may also take notice of the compactifications of the IIB theory in signature $(5,5)$, with conventional signs for the 3-form terms, that will give compact symmetry groups for suitable compactifications up to 5 dimensions, just like the M-theories in signatures $(5,6)$ and $(6,5)$.

In 7 dimensions transitions between 3 theories are possible if the subalgebra of $sl(5,\mathbb{R})$ relevant for the coset is $so(3,2)$. Note how its decompositions into $so(2,1)\oplus so(2)$, $so(2,1)\oplus so(1,1)$ and $so(3)\oplus so(2)$ reveal the signatures of the 3-tori and the 10 dimensional symmetry-group, and that in particular $so(3)\oplus so(1,1)$ is impossible. Similarly, $so(4,1)$ can only be decomposed in $so(3)\oplus so(1,1)$ and $so(2,1)\oplus so(2)$. Putting these facts together inevitably leads to table 10, that is confirmed by more sophisticated computation.

Table 11 collects the results of our computations for compactifications to 6 dimensions, where the global symmetry algebra is $so(5,5)$. For a more intuitive understanding of table 11, recall how the $so(4)$ symmetry of 4 compact directions is retrieved from $so(5)\oplus so(5)$. It is useful to proceed in two steps with the successive decompositions

$$\begin{aligned} so(5)\oplus so(5) &\to so(3)\oplus so(2)\oplus so(3)\oplus so(2) \\ &\to so(3)\oplus so(3)\oplus so(2)\cong so(4)\oplus so(2)\,, \end{aligned} \tag{7.6}$$

where on the second line, we have formed the diagonal algebra of the two $so(2)$ terms, and realized that $so(4)$ is not simple but consists of two $so(3)$ terms.

$\mathbb{R}_{m,n}$	$T_{p,q}$						h
	$++$	$+-$	$--$	$+-$	$++$	$--$	
$(0,6)$	$(1,3)$	$(1,3)$	$(3,1)$				$so(5,\mathbb{C})$
$(1,5)$	$(0,4)$						$so(5) \oplus so(5)$
$(1,5)$		$(0,4)$	$(2,2)$	$(0,4)$			$so(4,1) \oplus so(4,1)$
$(1,5)$					$(0,4)$		$so(5) \oplus so(5)$
$(2,4)$			$(1,3)$	$(3,1)$	$(3,1)$		$so(5,\mathbb{C})$
$(3,3)$			$(0,4)$	$(2,2)$	$(2,2)$	$(4,0)$	$so(3,2) \oplus so(3,2)$

Table 11: Dualities of IIB-theories in 6 dimensions.

$\mathbb{R}_{m,n}$	$T_{p,q}$						h
	$++$	$+-$	$--$	$+-$	$++$	$--$	
$(0,5)$	$(1,4)$	$(1,4)$	$(3,2)$	$(5,0)$			$sp(2,2)$
$(0,5)$					$(5,0)$		$sp(4)$
$(1,4)$	$(0,5)$						$sp(4)$
$(1,4)$		$(0,5)$	$(2,3)$	$(4,1)$	$(4,1)$		$sp(2,2)$
$(2,3)$			$(1,4)$	$(3,2)$	$(3,2)$	$(5,0)$	$sp(4,\mathbb{R})$

Table 12: Dualities of IIB-theories in 5 dimensions.

$\mathbb{R}_{m,n}$	$T_{p,q}$						h
	$++$	$+-$	$--$	$+-$	$++$	$--$	
$(0,4)$	$(1,5)$	$(1,5)$	$(3,3)$	$(5,1)$	$(5,1)$		$su^*(8)$
$(1,3)$	$(0,6)$						$su(8)$
$(1,3)$		$(0,6)$	$(2,4)$	$(4,2)$	$(4,2)$	$(0,6)$	$su(4,4)$
$(2,2)$			$(1,5)$	$(3,3)$	$(3,3)$	$(1,5)$	$sl(8,\mathbb{R})$

Table 13: Dualities of IIB-theories in 4 dimensions.

With the Lie algebra-isomorphisms $so(2,2) = so(2,1) \oplus so(2,1)$, and $so(3,\mathbb{C}) = sl(2,\mathbb{C}) = so(1,3)$, and the above embedding, the reader should be able to reconstruct table 11.

The relevant group theory for compactification of IIB-theory to 5 dimensions is not difficult. The global symmetry is $E_{6(6)}$. The group mixing 5 dimensions has algebra C_2, and has to embedded in the denominator algebra which is a real form of C_4. It is still feasible to realize these algebra's as matrix algebra's. To obtain the connection with the space-time signature, the isomorphisms $sp(2) \cong so(5)$, $sp(1,1) \cong so(4,1)$, and $sp(2,\mathbb{R}) = so(3,2)$ for the real forms of C_2 are useful. The reader less familiar with these algebra's can easily compute the signature and compare the real form with [25, 22], or another source listing real forms of the relevant algebra's

The algebra for 4 dimensions is somewhat similar to the one for 5 dimensions. The global symmetry algebra is $E_{7(7)}$. To extract the space-time signature from the denominator sub-algebra, we are dealing with real forms of A_3, to be embedded in a real form of A_7. The relevant real forms of A_3 are $su(4) \cong so(6)$, $su^*(4) = so(1,5)$, $su(2,2) = so(4,2)$,

$\mathbb{R}_{m,n}$	$T_{p,q}$						h
	$++$	$+-$	$--$	$+-$	$++$	$--$	
$(0,3)$	$(1,6)$	$(1,6)$	$(3,4)$	$(5,2)$	$(5,2)$	$(7,0)$	$so^*(16)$
$(1,2)$	$(0,7)$						$so(16)$
$(1,2)$		$(0,7)$	$(2,5)$	$(4,3)$	$(4,3)$	$(6,1)$	$so(8,8)$

Table 14: Dualities of IIB-theories in 3 dimensions.

$\mathbb{R}_{m,n}$	$T_{p,q}$							h
	$++$	$+-$	$--$	$+-$	$++$	$--$	$+-$	
$(0,2)$	$(1,7)$	$(1,7)$	$(3,5)$	$(5,3)$	$(5,3)$	$(1,7)$		H_9^{n1}
$(1,1)$	$(0,8)$							H_9^c
$(1,1)$		$(0,8)$	$(2,6)$	$(4,4)$	$(4,4)$	$(6,2)$	$(0,8)$	H_9^{n2}

Table 15: Dualities of IIB-theories in 2 dimensions.

$\mathbb{R}_{m,n}$	$T_{p,q}$								h
	$++$	$+-$	$--$	$+-$	$++$	$--$	$+-$	$++$	
$(1,0)$		$(0,9)$	$(2,7)$	$(4,5)$	$(4,5)$	$(6,3)$	$(1,8)$	$(1,8)$	H_{10}^{n1}
$(1,0)$	$(0,9)$								H_{10}^c

Table 16: Dualities of IIB-theories in 1 dimensions.

$sl(4,\mathbb{R}) = so(3,3)$. Again, those who are less comfortable with the matrix algebra's are reminded that also a direct computation is possible.

In 3 dimensions we have $E_{8(8)}$ as our global symmetry, while the local symmetry algebra must be a real form of $so(16,\mathbb{C})$ The embedding of $so(7)$ in $so(16)$ is straightforward. The easiest way to see it is via the chain of decompositions

$$so(16) \to so(7) \oplus so(7) \oplus so(2) \to so(7) \oplus so(2) \tag{7.7}$$

where in the last step again we have selected the diagonal sub-algebra from the 2 $so(7)$ algebra's. Although a number of entries in table 14 can be understood from this decomposition, constructing the full table requires more accuracy on various signs than this sketchy argument gives.

The denominator groups for compactification of IIB-theory to 2 dimensions, that are subgroups of the global symmetry algebra $E_{9(9)}$ appear in table 15. Note that the two possible non-compact algebra's H_9^{n1} corresponds to torus signatures where a there is an odd number of space as well as of time dimensions, while H_9^{n2} takes into account all signatures where there is an even number of space- and time directions (except for the usual IIB-theory on a euclidean 8-torus, that is covered by the compact form of H_9^c).

The table 16 contains our results for compactifications of IIB-theories to 1 dimension. These algebra's are sub-algebra's of $E_{10(10)}$. Again, as in the M-theory case this table hardly has any information (beyond the fact that the real form H_{10}^{n2} does not appear).

For compactification to 0 dimensions, or alternatively, any conjecture on the resurrection of IIB-theory from the E_{11} algebra [37] the relevant denominator algebra is unique. Computations along the lines of [18] reveal that it gives all the IIB theories presented in [3, 4], and no others.

8. Conclusions

In this paper we have extended and improved some techniques from [18]. These provide a firm mathematical framework, in which time-like compactifications of supergravities can be studied with relative ease. We have rederived and extended results of [16, 17, 3, 4] with the aim of elucidating the full duality web for the theories introduced in [3, 4].

The tables in this paper collect all maximal supergravity theories, and the dualities between their ultraviolet completions, the M- and type II-string theories. A remarkably simple formula (equation (2.28)) gives the groups that were previously determined by educated inspection and explicit reduction of higher dimensional theories. Moreover, our analysis has revealed the possibility of more groups: $\mathrm{SO}(3) \times \mathrm{SO}(1,1)$, $\mathrm{SO}(3,2) \times \mathrm{SO}(3,2)$, $\mathrm{Sp}(4, \mathbb{R})$, $\mathrm{SL}(8, \mathbb{R})$ can appear as denominator subgroups of $\mathrm{SL}(3, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$, $\mathrm{SO}(5,5)$, $E_{6(6)}$ and $E_{7(7)}$. And last but not least, it allows to do some computations with infinite dimensional groups, even though the understanding of these, both from the mathematical as well as from the supergravity viewpoint, is only rudimentary.

Even though we have restricted to $E_{n(n)}$ groups in the applications, most of the mathematical discussion was phrased in general terms or can easily be generalized to arbitrary groups. In appendix A we have computed the groups implied by the generalization of our formula (2.28) for arbitrary split groups. At least some of these should appear when considering time-like compactification of the theories in [20, 21], that are conjectured to be described by the symmetry algebra's in [36].

We expect that the developed formalism and insights from it may be useful to other problems, involving the algebraic structure of (super-)gravity. In particular their applicability to infinite-dimensional algebra's provides new tools for computation. We hope to report on these in the future.

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A. Possibilities for cosets

In the body of this paper we have restricted ourselves to cosets E_n/H_n relevant to compactifications of supergravity theories with maximal supersymmetry. There are however

G	$\mathcal{G}$	H
$A_{n(n)} = sl(n+1,\mathbb{R})$	$su(n+1-p,p)$	$so(n+1-p,p)$
$B_{n(n)} = so(n+1,n)$	$so(2n+1-2p,2p)$	$so(n+1-p,p) \oplus so(n-p,p)$
$C_{n(n)} = sp(n,\mathbb{R})$	$sp(n,\mathbb{R})$ $sp(n-p,p)$	$gl(n,\mathbb{R})$ $u(n-p,p)$
$D_{n(n)} = so(n,n)$	$so(2n-2p,2p)$ $so^*(2n)$	$so(n-p,p) \oplus so(n-p,p)$ $so(n,\mathbb{C})$
$E_{6(6)}$	$e_{6(-78)}$ $e_{6(-14)}$ $e_{6(2)}$	$sp(4)$ $sp(2,2)$ $sp(4,\mathbb{R})$
$E_{7(7)}$	$e_{7(-133)}$ $e_{7(-25)}$ $e_{7(-5)}$ $e_{7(7)}$	$su(8)$ $su^*(8)$ $su(4,4)$ $sl(8,\mathbb{R})$
$E_{8(8)}$	$e_{8(-248)}$ $e_{8(-24)}$ $e_{8(8)}$	$so(16)$ $so^*(16)$ $so(8,8)$
$F_{4(4)}$	$f_{4(-52)}$ $f_{4(-20)}$ $f_{4(4)}$	$sp(3) \oplus su(2)$ $sp(2,1) \oplus su(2)$ $sp(3,\mathbb{R}) \oplus sl(2,\mathbb{R})$
$G_{2(2)}$	$g_{2(-14)}$ $g_{2(2)}$	$so(4) \cong su(2) \oplus su(2)$ $so(2,2) \cong sl(2,\mathbb{R}) \oplus sl(2,\mathbb{R})$

Table 17: Possible cosets G/H for G split. H is defined as $\mathcal{G} \cap G$.

many more examples of theories involving sigma models on cosets G/H, coupled to gravity [26, 20, 21], that can appear in the dimensional reduction of various theories. One can extend the analysis for these theories to include reduction on one or more time-like directions. The methods of the present paper can be straightforwardly extended to these theories.

If we suppose that the group G is a split, finite dimensional simple Lie-group, then the generalization of formula (2.28) still applies. Let G be the split real form, and $\mathcal{G}$ be a real form of the complexification of G, generated by an inner involution. Then the possible denominator subgroups H can be easily characterized, by specifying the signature $\sigma(H)$. The possibilities for $\sigma(H)$ can be directly derived from the signature $\sigma(\mathcal{G})$ and rank $r(\mathcal{G})$ of the possible $\mathcal{G}$'s, by:

$$\sigma(H) = \frac{\sigma(\mathcal{G}) + r(\mathcal{G})}{2}\,. \tag{A.1}$$

Given a split G, the computation of the possible cosets is now a simple exercise, that can be easily carried out with the aid of [25], or the results collected in [21]. The answers are given in table 17.

We have also collected various results from our paper in this table for easy reference.

We stress again that in table 17 the only $\mathcal{G}$ that can appear are those generated by inner involutions; the reader will look in vain for entries such as $sl(n+1,\mathbb{R}), su^*(n+1)$ and other real forms that are generated by outer involutions. For the same reason we only listed $so(2n-2p,2p)$ and not $so(2n-2p-1,2p+1)$; the latter real forms are generated by involutions that are outer.

These cosets are relevant for reductions including time-like directions of the theories in [26, 20, 21, 22]. At least some of these cosets are inevitable contained in the theories listed in [36], but to decide which ones requires more detailed computation, that we will not perform here.

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